1MAE504 - Acoustics

C3-C4-C5 (Feb. 24, 2020)

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Here are the exercises and solutions of the C3-C4-C5 class of the module *Acoustics* 1MAE504.

Problem 1 What is the internal energy e

Solution The internal energy can be computed from thermodynamics. As the well known formula for the enthalpy $dH = C_p dT$, the internal energy is obtained through $de = c_v dT$, where $C_v = \frac{r}{\gamma - 1}$ and $dT = d(P/r\rho)$ from the ideal gas law. Combining both equation leads to $de = d\left(\frac{P}{\rho(\gamma-1)}\right)$, so that $e = \frac{P}{\rho(\gamma-1)}$.

Problem 2 Derive the linearized mass equation, assuming a quescient flow, i.e. $\vec{V}_0 = 0$.

Solution A Taylor expansion of the complete mass equation $\frac{\partial \rho}{\partial t} + div(\rho\vec{V}) = 0$ is performed. Variables are expanded at first order as $\rho = \rho_0 + \rho'$ and $\vec{V} = \vec{V}_0 + \vec{V}'$. Here, ρ_0 is assumed constant for the sake of simplicity. At 0th order, the expansion yields the equation for the mean flow, so that

$$\frac{\partial \rho_0}{\partial t} + div(\rho_0 \vec{V}_0) = 0 \tag{0.1}$$

At first order, injecting the previous equation as well as neglected the second order term $div(\rho'\vec{V}')$, it yields

$$\frac{\partial \rho'}{\partial t} + \rho_0 div(\vec{V}') = 0 \tag{0.2}$$

Problem 3 Derive the linearized momentum equation, assuming a quescient flow.

Solution As in the previous example, a first order Taylor expansion is performed. Since $\vec{V}_0 = \vec{0}$, the time-derivative is simply $\frac{\partial \rho_0 \vec{V}'}{\partial t}$. The convective derivative is second order at least, since involving product of ρ and $\vec{V}^2$. The RHS term is composed only on $-grad(P')$ (no viscosity). Consequently, the equation is

$$\frac{\partial \rho_0 \vec{V}'}{\partial t} + grad(P') = 0 \tag{0.3}$$

Problem 4 Derive the linearized energy equation.

Solution The energy equation is a 2nd order equation that can be obtained by combining the mass and momentum linearized equations. To do so, the trick is to use the multiplication rule for derivative $(f^2)' = 2f'f$: it allows us to create squared quantities inside time-derivative, typical of energy conservation laws.

Thus, the mass equation will be multiplied by p'/ρ_0 (noting that $p' = c_0^2 \rho'$, which gives:

$$\frac{1}{\rho_0 c_0^2} p' \frac{\partial p'}{\partial t} = \frac{1}{2\rho_0 c_0^2} \frac{\partial (p')^2}{\partial t} = -p' div(\vec{V}') \tag{0.4}$$

Then, the linearized momentum equation is multiplied by $\vec{V}'$, so that

$$\rho_0 \vec{V}' \frac{\partial \vec{V}'}{\partial t} = \frac{\partial (\rho_0 \vec{V}')^2/2}{\partial t} = -V' grad(p') \tag{0.5}$$

Now assembling the two equations as (0.4)+(0.5) leads to

$$\frac{\partial E}{\partial t} = -V' grad(p') - p' div(\vec{V}') \tag{0.6}$$

where $E = \frac{\rho_0}{2} (\vec{V}')^2 + \frac{1}{2\rho_0 c_0^2} (p')^2$ is the acoustic energy in a quescient flow.

The RHS term can be recast in a single term using the identity rule $div(f\vec{A}) = f div(\vec{A}) + \vec{A} \cdot grad(f)$, which therefore yields

$$\frac{\partial E}{\partial t} = -div(\vec{I}) \tag{0.7}$$

where $\vec{I} = p'\vec{V}'$ is the energy flux in a quescient flow, also called the acoustic intensity.

Problem 5 Show that the linearized acoustic energy implies a potential flow.

Solution The definition of a potential flow is $curl(\vec{V}') = \vec{0}$, which the implies that it exists a potential ϕ such that $\vec{V}' = grad(\phi)$.

Taking the curl of the momentum equation, assuming a null mean flow and constant denssity, yields:

$$\rho_0 \frac{\partial curl(\vec{V}')}{\partial t} = -curl(grad(p')) = \vec{0} \tag{0.8}$$

because for any function f, $curl(grad(f)) = \vec{0}$. Consequently, it show that $curl(V')$ is a constant, thus defined by the initial state. In general, it easy to show that an acoustic field at $t = 0$ is potential. It show in practice that acoustic waves cannot generate a rotational flow such as vortices, at least for small perturbations (linearization) and with our assumptions (e.g. no viscosity).

Problem 6 Obtain the acoustic wave equation

Solution Let's use the linearized mass (1) and momentum (2) equations. Taking $\frac{\partial (1)}{\partial t} - div(2)$ yields

$$\frac{\partial^2 \rho'}{\partial t^2} - \rho_0 div(\frac{\partial \vec{V}'}{\partial t}) = -\rho_0 div(\frac{\partial \vec{V}'}{\partial t}) + div(grad(p') \tag{0.9}$$

so that the 2nd and 3rd terms cancel out. The first term can be recast using the fluctuating pressure using $p' = c_0^2 \rho'$, and the last term is the definition of the Laplacian, since for any function f, $div(grad(f)) = \Delta f$. Consequently, the wave equation is obtained

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} = \Delta p' \tag{0.10}$$

Note that compare with the incompressible potential flow theory obtain for aerodynamic, here the equation is close, but not equal, to $\Delta p' = 0$. Indeed, since the flow is compressible, an additional term appears.

Problem 7 Obtain the Helmholtz equation

Solution This equation is obtained from the wave equation, also satisfied by the complex pressure $\tilde{p} = \hat{p}(x)e^{-j\omega t}$. Because of the linearity of the operator, the wave equation can be recast as using the classical derivation rule of the exponential:

$$\frac{1}{c_0^2}(-j\omega)^2\hat{p}(x)e^{-j\omega t} = \Delta(\hat{p}(x))e^{-j\omega t} \tag{0.11}$$

In this equation, the temporal term can be removed from both LHS and RHS term. Moreover, $\frac{1}{c_0^2}(-j\omega)^2 = -\left(\frac{\omega}{c_0}\right)^2 = -k^2$. Consequently, we obtained an equation for $\hat{p}(x)$, which now does not depend of the time, but only on the frequency ω (this equation will be solved in the frequency domain):

$$\Delta(\hat{p}(x)) + k^2\hat{p}(x) = 0 \tag{0.12}$$

Problem 8 Determine the frequencies and modal structures of acoustic waves in these 1D tube.

Solution In the lecture, we have seen that solutions of the 1D Helmholtz equation (complex form) can be written, either using the exponential or cos/sin basis, as:

$$\tilde{p}(x,t) = \left(Ae^{jkx} + Be-jkx)\right)e^{j\omega t} = \left(C\cos(kx) + D\sin(kx)\right)e^{j\omega t} \tag{0.13}$$

Here, because of the boundary conditions, we will chose the cos/sin expression, yet please note that this exercise can be solved using the exponential basis of course. Since one boundary condition is given by the acoustic velocity rather than the acoustic pressure, we first need to obtain the acoustic velocity expression. To do so, we can deduce the general solution for the velocity based on the pression expression and the linearized momentum equation, which leads to

$$\tilde{V}(x,t) = \frac{1}{j\rho_0 c_0}\left(C\sin(kx) - D\cos(kx)\right)e^{j\omega t} \tag{0.14}$$

For the first case, the boundary condition imposes that $\tilde{V}(x=0,t) = 0 = \frac{-De^{j\omega t}}{j\rho_0 c_0}$, so that $D = 0$ (this is why the cos/sin basis has been chosen, since it leads to a null constant, simplifying the expression).

The pressure now reads

$$\tilde{p}(x,t) = C\cos(kx)e^{j\omega t} \tag{0.15}$$

The second boundary condition is $\tilde{p}(x=L,t) = 0$. For that, either $C = 0$ (but it would lead to no acoustics in the tube), or $\cos(kL) = 0$. The solutions of this equation provides the specific frequencies of the acoustic modes in the tube : only these modes can "survive" in the tube to satisfy the boundary conditions (all other solutions would decay exponentially in time). Consequently, solutions are $kL = \pi/2$, $3\pi/2$, $5\pi/2$, etc., for which a general formula is $kL = (2n+1)\pi$. n is called the modal order.

The first mode $n = 0$ corresponds to $\omega = c_0\pi/2L$, so that $f = c_0/4L$. Note that $\lambda = c_0 T = c_0/f = 4L$: for that reasin, we call this mode a quarter-wave mode. The evolution of the mode in time is displayed in the following figure:

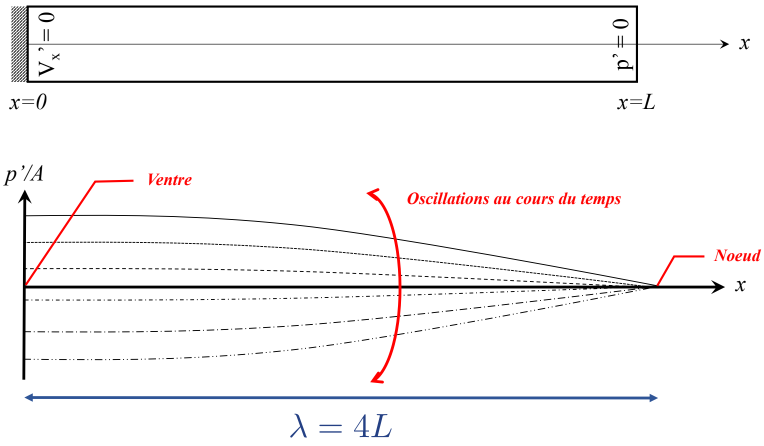


Figura 1: Steady quarter-wave mode, $\lambda = 4L$

For the second tube, the same approach is developed. However, the second boundary condition now specifies a null velocity (compares with a null pressure) at $x = L$. Consequently, the boundary condition reads $\tilde{V}(x = L, t) = 0 = \frac{C\sin(kL)}{j\rho_0 c_0} e^{j\omega t}$. Thus, now the specific frequencies (also called eigenfrequencies) are obtained by $\sin(kL) = 0$, for which solutions are $kL = n\pi$.

For the first mode, $n = 0$, we obtain a constant function which has to be null because of the boundary conditions. Consequently, the first acoustic mode is obtained for $n = 1$, leading to the frequency $f = c_0/2L$. The wavelength is $\lambda = c_0/f = 2L$: this mode is therefore called a half-wave mode. His modal structure evolving in time is displayed in the following figure:

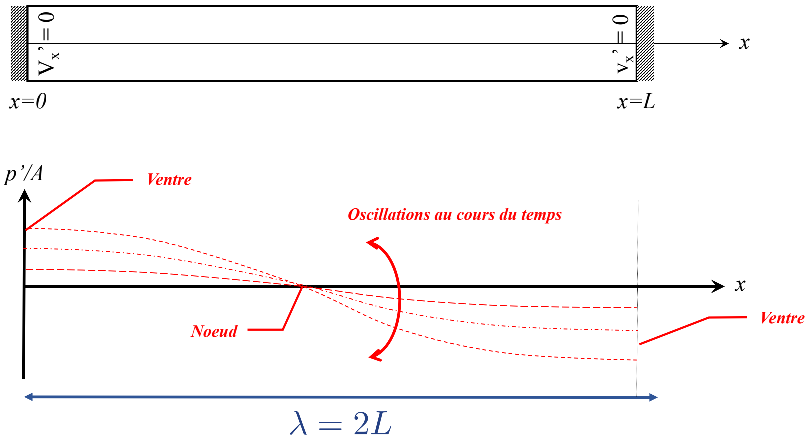


Figura 2: Steady half-wave mode, $\lambda = 4L$

The modes we obtained in these two tubes are called steady. As you can observe in the plot, the modal structure oscillates with some fixed points. The fixed points for which the acoustic pressure is always null is called a "pressure node". The same definition can be done for "velocity node". Note that other modes can exist: modes can be purely propagating, or have a mixed behavior between steady and propagating.

Problem 9 Acoustic modes in a disk

Solution Compared with the previous exercise where the solutions were known, here the only starting point is the wave equation written in polar coordinates, that we need to solve:

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \left(\frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} \right) = 0 \tag{0.16}$$

The method sol solve such an equation is the separation of variables, already seen in other courses (see for instance the potential flow around a cylinder). To do so, the solution is searched as:

$$p(r, \theta, t) = R(r)\Theta(\theta)T(t) \tag{0.17}$$

Once injected into the wave equation, it yields

$$\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} \right) + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{c_s^2}\frac{\ddot{T}}{T} \tag{0.18}$$

The RHS term depends only on the time, and the LHS only on the spatial coordinate, which suggests that LHS and RHS are constant functions. This constant is designated σ^2. Thus, it yields

$$\frac{1}{c_s^2}\frac{\ddot{T}}{T} = -\sigma^2 \tag{0.19}$$

The constant is searched as the opposite of a squared quantity so that only stable solutions are obtained. The same reasoning on the functions R and Θ leads to to constant (with constant $-m^2$, which reads:

$$\frac{\Theta''}{\Theta} = -m^2 \tag{0.20}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \sigma^2 r^2 = m^2 \tag{0.21}$$

The solutions of these equations are

$$\Theta(\theta) = a\sin(m\theta) + b\cos(m\theta) \tag{0.22}$$

$$R(r) = cJ_m(\sigma r) + dY_m(\sigma r) \tag{0.23}$$

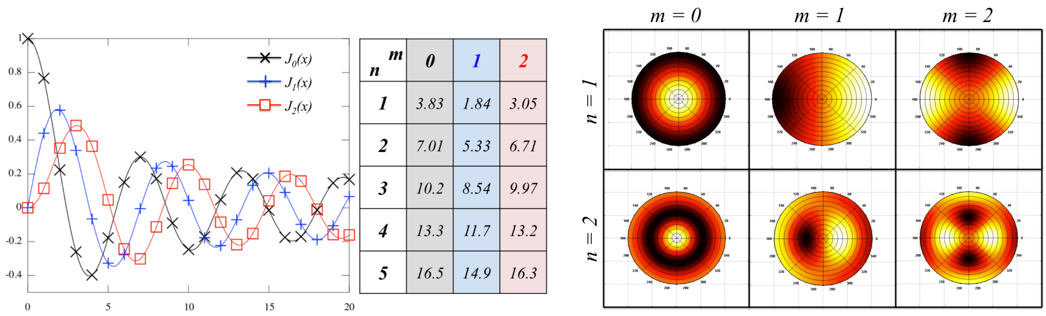
where J_m and Y_m are special functions, known as "Bessel functions.ºf the first and second kind. Note that the function Y_m goes to infinity when $r \to 0$, so that it cannot be a solution of the problem. It implies that $d = 0$.

The boundary condition at $r = R$, at the disk border, imposes $J'_m(\sigma R) = 0$.This is a similar equation than the one obtained for the 1D tube, but here it involves the special function J_m through its derivative. This function is well known in physics, so its zeros can be found in tables or any mathematical softwares (e.g. Matlab). Thus, this boundary condition implies discrete values for σ, which reads $\sigma_{m,n} = \chi_{m,n}/R$, where $\chi_{m,n}$ are the zeros of J'_m.

Similarly, the periodicity condition $p'(r, 0, t) = p'(r, 2\pi, t)$ imposes discrete values for m: m is called the azimuthal order of the mode. Note that the azimuthal modal order affects that function J_m, so the zeros of this function. These zeros are parametrized by an integer n, which will be the radial modal order". Finally, the angular pulsation of the mode (m, n) is given by:

$$\omega_0 = c_0\frac{\chi_{m,n}}{R} \tag{0.24}$$

Some results with various azimuthal and radial modal orders are displayed in the following figure, as well with the Bessel function:



n \ m	0	1	2
1	3.83	1.84	3.05
2	7.01	5.33	6.71
3	10.2	8.54	9.97
4	13.3	11.7	13.2
5	16.5	14.9	16.3

Figura 3: Bessel function, table of a few values of $\chi_{m,n}$, and modal structure for various (m, n) values.